thrust-to-weight ratio covered in Table 1 is that for nonescape from orbit ($T/mg < ½$), whereas the range covered in Table 2 is a portion of that for escape ($T/mg > ½$). In both tables, values of vehicle mass ratio obtained using Eq. (26) are listed for two values of specific impulse, 250 s and 500 s. The lower value is typical of those for solid rocket propellants, whereas the higher value is about the maximum attainable using the best liquid rocket propellants (hydrogen and oxygen). For a vehicle in circular orbit around Earth at a radial distance of 4100 miles (6,598 km), the value of $\sqrt{r_1^3/\mu}$ is 850 s.

There is a direct application of the solution presented herein to the problem of interplanetary transfer from the orbit of Earth to that of Mars using solar radiation as a means of propulsion.[5] Since solar radiation pressure varies inversely as the square of radial distance from the sun, as does the acceleration of gravity due to the sun, the value of T/mg is constant when the area of the solar sail and the total mass are fixed. Thus, results of the kind presented in Table 1 apply directly. However, since the orbit of Mars is considerably more eccentric than the nearly-circular orbit of Earth, the value of r_2/r_1 can range from a minimum of about 1.38 to a maximum of about 1.66, depending on the particular epoch. In the average case, the value of r_2/r_1 is about 1.52, so that $\bar{V}_2^2 = 0.65789$, $T/mg = 0.17$, and $\sqrt{\mu/r_1^3}\Delta t = 4.8750$. For a vehicle in circular orbit around the sun at a radial distance equal to that of Earth (93,000,000 miles or about 150,000,000 km), the value of $\sqrt{r_1^3/\mu}$ is 5.0298×10^6 s or 58.215 days. The travel time, therefore, from the orbit of Earth to the midpoint (between aphelion and perihelion) on the orbit of Mars using solar radiation pressure for propulsion is found to be 58.215 d $\times$ 4.8750 = 283.8 days.

Conclusions

An analytic solution has been obtained for the effects of continuous radial (vertical) thrust on the orbital motion and mass loss of a vehicle initially in a circular orbit. It is found that, with continuous application of radial thrust equal in magnitude to a fixed fraction of the vehicle weight (as the product of vehicle mass and the ambient acceleration of gravity), the flight path can take one of two possible forms. If the value of thrust-to-weight ratio is greater than one-half, escape speed will eventually be reached along an unwinding spiral trajectory. If the value of this ratio is less than one-half, the vehicle will simply spiral out to a maximum altitude (apogee) and then return along a symmetrical trajectory to its initial position at the inception of thrusting. When the value of this ratio is one-half, the spiral path to maximum altitude extends to infinity. Formulas have been found for the orbital motion and time of flight along each trajectory and for mass loss due to expenditure of rocket propellant (based on the specific impulse of the propulsion system).

Appendix: Derivation of Angular Rotation

The flight-path angle γ is given by

$$\tan\gamma = \frac{dr}{r\,d\theta} \tag{A1}$$

with θ being the angle of rotation about the planet so that

$$d\theta = \frac{\cos\gamma}{\sin\gamma}\frac{dr}{r} \tag{A2}$$

and, consequently, using Eqs. (8) and (13) with $\bar{V}_1 = 1$ and $\gamma_1 = 0$,

$$d\theta = 2\bar{V}\left[\bar{V}^2(\bar{V}^2 + b)/(1+b) - 1\right]^{-1/2}\frac{d\bar{V}}{(\bar{V}^2 + b)} \tag{A3}$$

where $b = 2(T/mg - 1)$. Then, letting $x = \bar{V}^2$ so that $dx = 2\bar{V}\,d\bar{V}$, Eq. (A3) becomes

$$d\theta = \left[x(x+b)/(1+b) - 1\right]^{-1/2}\frac{dx}{x+b} \tag{A4}$$

which in turn, letting $y = x + b$ so that $dy = dx$, becomes

$$d\theta = \left[y(y-b)/(1+b) - 1\right]^{-1/2}\frac{dy}{y}$$

$$= \left[\frac{y^2}{(1+b)} - \frac{by}{(1+b)} - 1\right]^{-1/2}\frac{dy}{y} \tag{A5}$$

This equation may be integrated according to a standard formula[6] to obtain

$$\theta = \sin^{-1}\left[\frac{-by/(1+b) - 2}{|y|\,[b^2/(1+b)^2 + 4/(1+b)]^{1/2}}\right] \tag{A6}$$

which, after applying the limits of integration, $y_1 = 1 + b$ and $y_2 = \bar{V}_2^2 + b$, yields finally

$$\theta_2 - \theta_1 \equiv \Delta\theta = \pi/2$$

$$- \sin^{-1}\left\{\frac{(\bar{V}_2^2 + b)b/(1+b) + 2}{|\bar{V}_2^2 + b|\,[b^2/(1+b)^2 + 4/(1+b)^{1/2}}\right\} \tag{A7}$$

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Trajectory Design for Robotic Manipulators in Space Applications

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Introduction

THIS paper presents a trajectory design method for space robots that has been developed by the author. A subsequent paper that applies the technique to a practical robot has already been published.[1]

Robotic manipulators used in space applications are expected to operate under microgravity conditions.[2,3] Base reactions of a space manipulator are directly exerted on the supporting space structure, which would be typically a space vehicle (e.g., space shuttle) or a space station. It is desirable to

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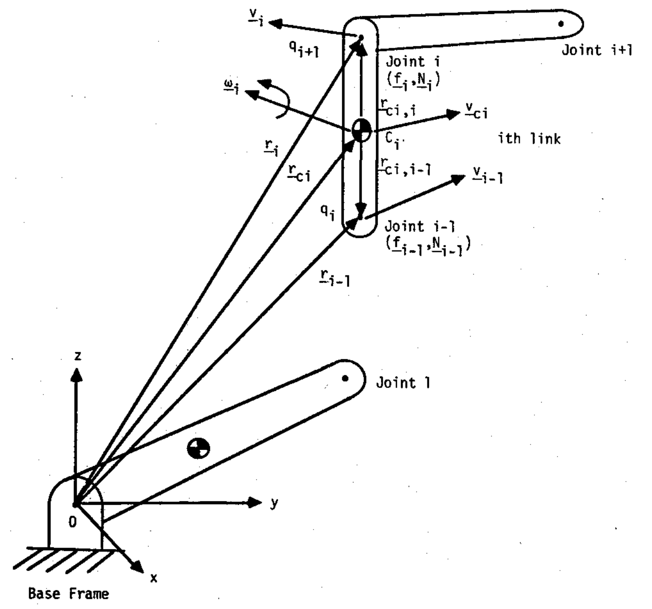

Fig. 1 Link nomenclature.

make these reactions as small as possible in order to reduce their influence on the dynamics of the supporting space structure. This will help in controlling the space structure, which is a crucial consideration in its own right, in space applications. Furthermore, in delicate experiments conducted in space, the test specimen would have to be moved carefully without subjecting it to excessive accelerations and jerks, but at reasonably high speeds. It follows that minimization of base reactions and limitation of end-effector accelerations and jerks are important performance objectives for space manipulators.

Dynamics

The method of joint trajectory generation developed in this paper consists of two steps: 1) plan the end-effector trajectory to satisfy the motion requirements of the pay load, and 2) minimize the base reactions using redundant kinematics of the manipulator, subject to the end-effector trajectory developed in step 1. Given an end-effector trajectory, the objective in step 2 would be to determine the joint trajectories that will minimize a suitable cost function in base reactions. Equations for base reaction forces and moments expressed in terms of the joint trajectories are needed here.

Consider ith link of an n-link manipulator as shown in Fig. 1. The Newton-Euler equations for this link consist of the force-momentum equations,

$$f_{i-1} - f_i + m_i g = \frac{d}{dt}(m_i v_{ci}) \tag{1}$$

and the moment-angular momentum equations about the centroid C_i of the link,

$$N_{i-1} - N_i + r_{ci,i-1} \times f_{i-1} - r_{ci,i} \times f_i = \frac{d}{dt}(I_{ci}\omega_i) \tag{2}$$

Here, f_{i-1} is the force vector at the i-1th joint of the ith link, N_{i-1} the moment (torque) vector at the i-1th joint of the ith link, g the vector representing acceleration due to gravity, m_i the mass of the ith link, I_{ci} the moment of inertia matrix of the ith link about C_i, v_{ci} the velocity of the centroid of the ith link, ω_i the angular velocity of the ith link, and $r_{a,b}$ the position vector from point a to point b.

All vectors are expressed in the base frame. The forces and the moments at the ith joint of the ith link are $-f_i$ and $-N_i$, respectively, as dictated by Newton's third law.

Next, by substituting

$$r_{ci,i-1} = r_{i-1} - r_{ci}, \text{ and } r_{ci,i} = r_i - r_{ci}$$

in Eq. (2), and on using Eq. (1), one obtains

$$N_{i-1} - N_i + r_{i-1} \times f_{i-1} - r_i \times f_i - r_{ci} \times \left[\frac{d}{dt}(m_i v_{ci}) - m_i g\right] = \frac{d}{dt}(I_{ci}\omega_i) \tag{3}$$

It is clear that the inertia matrix I_{ci} is constant with respect to a body frame fixed to the ith link. Accordingly, since this body frame has an angular velocity ω_i, we have

$$\frac{d}{dt}(I_{ci}\omega_i) = I_{ci}\dot{\omega}_i + \omega_i \times (I_{ci}\omega_i) \tag{4}$$

By substituting Eq. (4) and (3), one gets

$$N_{i-1} - N_i + r_{i-1} \times f_{i-1} - R_i \times f_i - m_i r_{ci} \times (\dot{v}_{ci} - g) = I_{ci}\dot{\omega}_i + \omega_i \times (I_{ci}\omega_i) \tag{5}$$

For the n links of the manipulator there are n equations in Eq. (1) and n equations in Eq. (5). An equation for the base reaction force is obtained by summing the n equations given by Eq. (1); thus,

$$f_o = \sum_{i=1}^{n} m_i(\dot{v}_{ci} - g) \tag{6}$$

Similarly, by summing the n equations in Eq. (5) and using the fact that $r_o = 0$ (see Fig. 1), the equation for the base reaction moment is obtained; thus,

$$N_o = \sum_{i=1}^{n} [I_{ci}\dot{\omega}_i + \omega_i \times (I_{ci}\omega_i) + m_i r_{ci} \times (\dot{v}_{ci} - g)] \tag{7}$$

For the present purposes Eqs. (6) and (7) have to be expressed in terms of joint trajectories. The required kinematic relations are as follows:

Angular velocities

$$\omega_1 = \dot{q}_1 \quad \text{and} \quad \omega_i = \omega_{i-1} + \dot{q}_i \tag{8}$$

Angular accelerations

$$\dot{\omega}_1 = \ddot{q}_1 \quad \text{and} \quad \dot{\omega}_i = \dot{\omega}_{i-1} + \ddot{q}_i + \omega_{i-1} \times \dot{q}_i \tag{9}$$

Note that the scalars $\dot{q}_i$ and $\ddot{q}_i$ are joint angular velocities and joint angular accelerations, which are "relative" variables. They must be expressed as vectors $\dot{q}_i$ and $\ddot{q}_i$ in the base frame, when using Eqs. (8) and (9).

Rectilinear velocities:

$$v_o = 0, \quad v_{ci} = v_{i-1} + \omega_i \times r_{i-1,ci}$$
$$v_i = v_{i-1} + \omega_i \times r_{i-1,i} \tag{10}$$

Rectilinear accelerations:

$$a_o = 0$$
$$\dot{v}_{ci} = a_{i-1} + \dot{\omega}_i \times r_{i-1,ci} + \omega_i \times (\omega_i \times r_{i-1,ci})$$
$$a_i = a_{i-1} + \dot{\omega}_i \times r_{i-1,i} + \omega_i \times (\omega_i \times r_{i-1,i}) \tag{11}$$

Equations (6-11) express the base reactions in terms of q, $\dot{q}$, and $\ddot{q}$. Note that q does not explicitly appear in these equations but is present in the vectors r_{ci}, $r_{i-1,ci}$ and $r_{i-1,i}$ through the coordinate transformations that are necessary to express these vectors in the base frame.[4]

Base Reaction Minimization

A suitable measure for the base reaction would be the weighted quadratic cost function

$$J = \boldsymbol{R}^T \boldsymbol{Q} \boldsymbol{R} \tag{12}$$

in which the reaction vector $\boldsymbol{R}$ is given by

$$\boldsymbol{R} = \begin{bmatrix} f_o \\ N_o \end{bmatrix} \tag{13}$$

and $\boldsymbol{Q}$ is a positive-definite (typically diagonal) weighting matrix that takes into consideration not only the relative importance of each forcing component but also the dimensional incompatibility that exists between f_o and N_o.

An alternative cost function would be the time integral of J over the trajectory duration.[5] This alone is not acceptable in the present problem because it would not appropriately include the severity of short-term impulsive-type base reactions. Instead, the proposal is to segment the entire trajectory into small steps and minimize J [Eq. (12)] during each step.

Suppose the end-effector trajectory is expressed in the usual manner as a sixth-order vector $\boldsymbol{v}(t)$ containing the position (three elements) and orientation (three elements) of a body frame attached to the end effector. The incremental joint motion $\delta \boldsymbol{q}$ that would be necessary to provide a required increment $\delta \boldsymbol{y}$ in the end-effector motion, is given by

$$\delta \boldsymbol{y} = \boldsymbol{J} \delta \boldsymbol{q} \tag{14}$$

in which $\boldsymbol{J}$ denotes the $6 \times n$ Jacobian matrix of the manipulator.[4] With redundant kinematics we have $n > 6$. The velocity relationship corresponding to Eq. (14) is

$$\boldsymbol{v} = \boldsymbol{J} \dot{\boldsymbol{q}} \tag{15}$$

and the acceleration relationship is

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} \dot{\boldsymbol{q}} + \boldsymbol{J} \ddot{\boldsymbol{q}} \tag{16}$$

in which $\boldsymbol{v} = \dot{\boldsymbol{y}} =$ end-effector velocity and $\boldsymbol{a} = \dot{\boldsymbol{v}} =$ end-effector acceleration. Note that Eq. (15) can be written in the index form $v_i = J_{ij} \dot{q}_j$ that, when differentiated becomes $a_i = (\mathrm{d}/\mathrm{d}t)(J_{ij}) \dot{q}_j + J_{ij} \ddot{q}_j$, which is indeed equivalent to Eq. (16).

In utilizing the redundant kinematics to minimize Eq. (12), suppose that

$$\operatorname{rank} \boldsymbol{J} = 6 \tag{17}$$

Also, without loss of generality, suppose that the first six columns of $\boldsymbol{J}$ are linearly independent. This simply depends on the order in which the joint coordinates are arranged in $\boldsymbol{q}$. Accordingly, as in the generalized inverse method given in Ref. 6, the Jacobian matrix can be partitioned into

$$\boldsymbol{J} = [\boldsymbol{J}_n \, \boldsymbol{J}_r] \tag{18}$$

in which $\boldsymbol{J}_n$ is a nonsingular 6×6 matrix corresponding to a set of independent joint coordinates $\boldsymbol{q}_n$, and $\boldsymbol{J}_r$ is a $6 \times (n-6)$ submatrix corresponding to a particular choice of redundant joint coordinates $\boldsymbol{q}_r$. Similarly, the vector $\boldsymbol{q}$ is partitioned according to

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_n \\ \boldsymbol{q}_r \end{bmatrix} \tag{19}$$

Consequently, Eqs. (14–16) become

$$\delta \boldsymbol{y} = \boldsymbol{J}_n \delta \boldsymbol{q}_n + \boldsymbol{J}_r \delta \boldsymbol{q}_r \tag{20}$$

$$\boldsymbol{v} = \boldsymbol{J}_n \dot{\boldsymbol{q}}_n + \boldsymbol{J}_r \dot{\boldsymbol{q}}_r \tag{21}$$

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{J}_n}{\mathrm{d}t} \dot{\boldsymbol{q}}_n + \frac{\mathrm{d}\boldsymbol{J}_r}{\mathrm{d}t} \dot{\boldsymbol{q}}_r + \boldsymbol{J}_n \ddot{\boldsymbol{q}}_n + \boldsymbol{J}_r \ddot{\boldsymbol{q}}_r \tag{22}$$

Since $\boldsymbol{J}_n$ is nonsingular, Eqs. (20–22) can be used to express the nonredundant joint motion variables in terms of the redundant joint motion variables

$$\delta \boldsymbol{q}_n = \boldsymbol{J}_n^{-1} \delta \boldsymbol{y} - \boldsymbol{J}_n^{-1} \boldsymbol{J}_r \delta \boldsymbol{q}_r \tag{23}$$

$$\dot{\boldsymbol{q}}_n = \boldsymbol{J}_n^{-1} \boldsymbol{v} - \boldsymbol{J}_n^{-1} \boldsymbol{J}_r \dot{\boldsymbol{q}}_r \tag{24}$$

$$\ddot{\boldsymbol{q}}_n = \boldsymbol{J}_n^{-1} \boldsymbol{a} - \boldsymbol{J}_n^{-1} \frac{\mathrm{d}\boldsymbol{J}_n}{\mathrm{d}t} \boldsymbol{J}_n^{-1} \boldsymbol{v} - \boldsymbol{J}_n^{-1} \boldsymbol{J}_r \ddot{\boldsymbol{q}}_r + \boldsymbol{J}_n^{-1} \left[\frac{\mathrm{d}\boldsymbol{J}_n}{\mathrm{d}t} \boldsymbol{J}_n^{-1} \boldsymbol{J}_r - \frac{\mathrm{d}\boldsymbol{J}_r}{\mathrm{d}t} \right] \dot{\boldsymbol{q}}_r \tag{25}$$

The derivatives $\mathrm{d}\boldsymbol{J}_n/\mathrm{d}t$ and $\mathrm{d}\boldsymbol{J}_r/\mathrm{d}t$ remain to be expressed in terms of $\dot{\boldsymbol{q}}_r$. To accomplish this, suppose that

$$\boldsymbol{L} = \frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \left[\frac{\mathrm{d}\boldsymbol{J}_n}{\mathrm{d}t} \, \frac{\mathrm{d}\boldsymbol{J}_r}{\mathrm{d}t} \right] \tag{26}$$

Consider the general element L_{ij} in matrix $\boldsymbol{L}$. This can be expressed as

$$L_{ij} = \frac{\mathrm{d}}{\mathrm{d}t} J_{ij} = \sum_{k=1}^{n} \frac{\partial J_{ij}}{\partial q_k} \dot{q}_k = (\Delta_q J_{ij})^T \dot{\boldsymbol{q}}$$

in which the gradient operator

$$\Delta_q = \left[\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots, \frac{\partial}{\partial q_n} \right]^T = \left[\Delta_{qn} \, \Delta_{qr} \right]^T$$

Note that

$$\Delta_{qn} = \left[\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots, \frac{\partial}{\partial q_6} \right]^T \tag{27}$$

$$\Delta_{qr} = \left[\frac{\partial}{\partial q_7}, \frac{\partial}{\partial q_8}, \ldots, \frac{\partial}{\partial q_n} \right]^T \tag{28}$$

It follows that $L_{ij} = (\Delta_{qn} J_{ij})^T \dot{\boldsymbol{q}}_n + (\Delta_{qr} J_{ij})^T \dot{\boldsymbol{q}}_r$ which, in view of Eq. (24), becomes

$$L_{ij} = (\Delta_{qn} J_{ij})^T \boldsymbol{J}_n^{-1} \boldsymbol{v} + [(\Delta_{qr} J_{ij})^T - (\Delta_{qn} J_{ij})^T \boldsymbol{J}_n^{-1} \boldsymbol{J}_r] \dot{\boldsymbol{q}}_r \tag{29}$$

Optimization Scheme

Suppose that the end-effector trajectory is divided into a number of steps. For convenience, assume that the duration of each step is the same even though this assumption is not essential to the scheme to be described.

First, it is necessary to express the cost function Eq. (12) in terms of a set of independent parameters that could be chosen to minimize J. A desirable option would be to use $\delta \boldsymbol{q}_r$ for this purpose, in each time step. But the dependence of $\boldsymbol{q}_r$, $\dot{\boldsymbol{q}}_r$, and $\ddot{\boldsymbol{q}}_r$ on a particular choice of $\delta \boldsymbol{q}_r$ should be taken into account. Of course, the dependence of $\delta \boldsymbol{q}_n$, $\dot{\boldsymbol{q}}_n$, and $\ddot{\boldsymbol{q}}_n$ on these parameters is explicitly included in the analysis through Eqs. (23–25) along with Eq. (29). The joint trajectory at an arbitrary time t within an arbitrary step may be expressed as $\boldsymbol{q} = \boldsymbol{q}_o + [\delta \boldsymbol{q}_n \, \delta \boldsymbol{q}_r]^T$, in which $\boldsymbol{q}_o$ is the value of $\boldsymbol{q}$ at the start of the step.

Now, substituting Eq. (23), one obtains

$$\boldsymbol{q} = \boldsymbol{q}_o + \begin{bmatrix} \boldsymbol{J}_n^{-1} \\ 0 \end{bmatrix}_0 \delta \boldsymbol{y} + \begin{bmatrix} -\boldsymbol{J}_n^{-1} \boldsymbol{J}_r \\ \boldsymbol{1} \end{bmatrix}_0 \delta \boldsymbol{q}_r \tag{30}$$

Note that the subscript zero identifies quantities evaluated at the known starting point $(\boldsymbol{q}_0)$ of the trajectory step, and $\boldsymbol{1}$ denotes the identity matrix of size $r \times r$.

In the present optimization scheme, the parameters of optimization are the elements of a $(n-6) \times m$ matrix C that satisfy the equation

$$\dot{q}_r = C f(t) \tag{31}$$

in which $f(t)$ is a known set of (m) shape functions that is used to represent the variation of $\dot{q}_r$ within a trajectory step. Then,

$$\ddot{q}_r = C \dot{f}(t) \tag{32}$$

Furthermore, Eq. (30) can be expressed as

$$q = q_o + \begin{bmatrix} J_n^{-1} \\ 0 \end{bmatrix}_0 \delta y + \begin{bmatrix} -J_n^{-1} J_r \\ 1 \end{bmatrix}_0 C f(t)\, \delta t \tag{33}$$

Without loss of generality t may be set to zero at the start of each step.

Using Eqs. (31-33) it is possible to express the cost function Eq. (12) in terms of the unknown coefficient matrix C. Then, the optimization scheme would proceed as follows: 1) Divide the end-effector trajectory $y(t)$ into a sufficiently large number of segments; 2) for a known initial configuration of the manipulator, obtain C that will minimize the cost function J at the end point of the segment; 3) compute the joint trajectory over the segment using Eq. (33), base reactions using Eqs. (6) and (7), and the cost function using Eq. (12); and 4) go to the next segment of the trajectory and repeat steps 2 and 3. Stop if the end of the trajectory is reached.

Selection of the end-effector trajectory is discussed in the next section. Choice of the shaping function vector $f(t)$ deserves some discussion. An appropriate choice would be

$$f(t) = [1, t, t^2/2!, \ldots, t^{m-1}/(m-1)!]^T \tag{34}$$

With this choice, the first three columns of the C matrix become the joint velocity vector, the joint acceleration vector, and the joint jerk vector, respectively, at the start of the present trajectory segment. Another option would be to employ trigonometric functions in $f(t)$, as in Rayleigh-Ritz techniques.[5] As long as the trajectory segments are sufficiently small and/or size of C is sufficiently large, any reasonable choice for $f(t)$ should produce satisfactory results.

At the starting point of the trajectory, the manipulator configuration is not unique, due to the presence of redundant degrees of freedom. It might be possible to use this redundancy in the first segment of optimization to further reduce the base reactions. One way to handle this is to employ the joint coordinates of the starting configuration of the manipulator as an additional set of optimization parameters, subject to the constraints given by the (nonlinear) kinematic equations of the manipulator for the starting position and orientation of the end effector.

End-Effector Trajectory Generation

A convenient method of limiting acceleration and jerk of the end-effector trajectory in space-based delicate experiments, is to employ curtate-cycloidal motions.[7] Consider the motion from a specified starting position and orientation to a specified end position and orientation of the end effector. Barring work-space constraints, a simple way to execute the corresponding motion is to move the origin of a body frame fixed to the end effector, along a straight line and continuously rotate this frame with respect to the base frame so that the required final orientation is reached at the end point of the path.

Suppose that the position along the straight-line path of the end effector is denoted by y. It is possible to change the orientation of the end-effector frame using a rotation θ about an axis w fixed in direction with respect to the base frame. The straight-line path and the axis of rotation w can be established beforehand, once the initial and final positions and orientations are known. Then the end-effector trajectory would be completely defined by $y(t)$ and $\theta(t)$. Both $y(t)$ and $\theta(t)$ are varied according to curtate cycloids and are synchronized. Specifically, the velocity $v = \dot{y}$ and the angular velocity $\omega = \dot{\theta}$ are defined in the parametric form by

$$v(p) = b_1 (1 - \cos p) \tag{35}$$

$$\omega(p) = b_2 (1 - \cos p) \tag{36}$$

$$t(p) = a(p - c \sin p) \tag{37}$$

$$a, b_1, b_2 > 0, \quad 1 > c > 0, \quad \text{and} \quad 2\pi \ge p \ge 0$$

in which p is a parameter that defines the evolution of time t during motion. The starting point ($p = 0$) and the end point ($p = 2\pi$) are assigned zero velocities and zero angular velocities. The duration of excursion is given by

$$T = 2\pi a \tag{38}$$

Without loss of generality, starting values of y and θ can be assumed zero. The position and orientation at time t are obtained by integrating Eqs. (35) and (36) using these initial conditions. This gives

$$y = ab_1\{[1 + (c/2)]p - (c+1)\sin p + (c/4)\sin 2p\} \tag{39}$$

$$\theta = ab_2\{[1 + (c/2)]p - (c+1)\sin p + (c/4)\sin 2p\} \tag{40}$$

Note that the final position and orientation are given by

$$y(T) = ab_1[1 + (c/2)]\, 2\pi \tag{41}$$

$$\theta(T) = ab_2[1 + (c/2)]\, 2\pi \tag{42}$$

Linear and angular accelerations a and α are obtained by differentiating Eqs. (35) and (36); thus,

$$a(p) = \frac{b_1 \sin p}{a(1 - c\cos p)} \tag{43}$$

$$\alpha(p) = \frac{b_2 \sin p}{a(1 - c\cos p)} \tag{44}$$

The maximum acceleration values are

$$|\dot{a}_{\max}| = \frac{b_1}{a\sqrt{1-c^2}} \tag{45}$$

$$|\alpha_{\max}| = \frac{b_2}{a\sqrt{1-c^2}} \tag{46}$$

and both occur at the same instant given by

$$p_{a\max} = \cos^{-1} c \tag{47}$$

Linear and angular jerks $\dot{a}$ and $\dot{\alpha}$ are given by

$$\dot{a}(p) = \frac{b_1}{a_2} \frac{(\cos p - c)}{(1 - c\cos p)^3} \tag{48}$$

$$\dot{\alpha}(p) = \frac{b_2}{a_2} \frac{(\cos p - c)}{(1 - c\cos p)^3} \tag{49}$$

These jerks remain finite throughout the trajectory as desired. There are four stationary values for each jerk expression. The largest magnitudes, however, would be given by

$$|\dot{a}|_{\max} = \frac{b_1}{a^2(1-c)^2} \tag{50}$$

$$|\dot{\alpha}|_{\max} = \frac{b_2}{a^2(1-c)^2} \tag{51}$$

These occur at the end points of the trajectory ($p = 0$ and $p = 2\pi$).

Once the end-effector trajectory is defined in this manner by Eqs. (39) and (40), it is a straightforward task to express the vectors $\boldsymbol{y}$, $\boldsymbol{v}$, and $\boldsymbol{a}$ that are needed in Eqs. (23-25), (29), and (33), for example. Specifically, if the unit vector along the end-effector path is $\boldsymbol{u}$ and the unit vector representing the fixed axis about which the rotation θ is made is $\boldsymbol{w}$, both expressed in the base frame, then

$$\boldsymbol{y} = \begin{bmatrix} y & \boldsymbol{u} \\ \theta & \boldsymbol{w} \end{bmatrix} \tag{52}$$

$$\boldsymbol{v} = \begin{bmatrix} v & \boldsymbol{u} \\ \omega & \boldsymbol{w} \end{bmatrix} \tag{53}$$

$$\boldsymbol{a} = \begin{bmatrix} a & \boldsymbol{u} \\ \alpha & \boldsymbol{w} \end{bmatrix} \tag{54}$$

in which y, θ, v, ω, a, and α are as expressed by Eqs. (39), (40), (35), (36), (43), and (44).

Conclusions

Design of the end-effector trajectory of a robotic manipulator according to motion specifications such as acceleration and jerk limits, and the design of the joint trajectories to minimize base reactions have been developed in this paper. The techniques presented have potential applications in space manipulators. Curtate cycloids were used to represent the end-effector motion. These have the benefits of finite acceleration and finite jerk. Kinematic redundancy is exploited in designing joint trajectories that will minimize a quadratic cost function in base reaction components. An application of the technique is given in Ref. 1.

The question of whether the corresponding results are global will be addressed in future work. Also, the effect of using alternative cost functions, such as a quadratic integral function, needs to be studied. In this paper, we have not addressed the problem of whether the actuators are capable of generating the torques required by the trajectory.[8] The sensitivity of the weighting matrix in emphasizing certain reaction components is being investigated.[8] In the example given in this paper, polynomial shape functions were used to represent redundant-joint trajectories. The potential advantages and possible disadvantages of using other types of shape functions should be studied as well.

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Jacobi Method for Unsymmetric Eigenproblems

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Introduction

THE complete solution of the eigenvalue problem via the Jacobi method is widely used in symmetric problems. Moreover, where the problem can be stated as the product of two symmetric matrices, as in problems of dynamics, a solution consisting of two Jacobi solutions is common practice. This paper offers a complete solution of the unsymmetric problem via a variation of the Jacobi method. The eigenvalues emerge in algebraic order, and there is no restriction of symmetry in the problem. Proof of convergence is included.

In the matric expression

$$AV = VD \tag{1}$$

V and D are the matrices of eigenvectors and eigenvalues of the nondegenerate matrix A. Of course, D is a diagonal matrix.

Matrices V and D may be found by way of matrices X, an orthogonal matrix, and Λ, a triangular matrix that contains the eigenvalues of A on its diagonal, such that

$$AX = X\Lambda \tag{2}$$

Let Y be the matrix of eigenvectors of Λ. Then,

$$\Lambda Y = YD \tag{3}$$

or

$$\Lambda = YDY^{-1} \tag{4}$$

Equations (2) and (4) may be combined to get

$$AX = XYDY^{-1} \tag{5}$$

or

$$AXY = XYD \tag{6}$$

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